

Extension of Summation Formulas involving Stirling series

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Abstract

This paper presents a family of rapidly convergent summation formulas for various finite sums of the form $\sum_{k=0}^{\lfloor x \rfloor} f(k)$, where x is a positive real number.

1 Introduction

In this paper we will use the Euler-Maclaurin summation formula [3, 5] to obtain rapidly convergent series expansions for finite sums involving Stirling series [1]. Our key tool will be the so called Weniger transformation [1].

For example, one of our summation formulas for the sum $\sum_{k=0}^{\lfloor x \rfloor} \sqrt{k}$, where $x \in \mathbb{R}^+$ is

$$\begin{aligned} \sum_{k=0}^{\lfloor x \rfloor} \sqrt{k} &= \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4\pi}\zeta\left(\frac{3}{2}\right) - \sqrt{x}B_1(\{x\}) + \sqrt{x} \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^k (-1)^l \frac{(2l-3)!!}{2^l(l+1)!} S_k^{(1)}(l) B_{l+1}(\{x\})}{(x+1)(x+2)\cdots(x+k)} \\ &= \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4\pi}\zeta\left(\frac{3}{2}\right) + \left(\frac{1}{2} - \{x\}\right) \sqrt{x} + \frac{\left(\frac{1}{4}\{x\}^2 - \frac{1}{4}\{x\} + \frac{1}{24}\right) \sqrt{x}}{(x+1)} \\ &\quad + \frac{\left(\frac{1}{24}\{x\}^3 + \frac{3}{16}\{x\}^2 - \frac{11}{48}\{x\} + \frac{1}{24}\right) \sqrt{x}}{(x+1)(x+2)} + \frac{\left(\frac{1}{64}\{x\}^4 + \frac{3}{32}\{x\}^3 + \frac{21}{64}\{x\}^2 - \frac{7}{16}\{x\} + \frac{53}{640}\right) \sqrt{x}}{(x+1)(x+2)(x+3)} \\ &\quad + \frac{\left(\frac{1}{128}\{x\}^5 + \frac{19}{256}\{x\}^4 + \frac{109}{384}\{x\}^3 + \frac{29}{32}\{x\}^2 - \frac{977}{768}\{x\} + \frac{79}{320}\right) \sqrt{x}}{(x+1)(x+2)(x+3)(x+4)} + \dots, \end{aligned}$$

where the $B_l(x)$'s are the Bernoulli polynomials and $S_k^{(1)}(l)$ denotes the Stirling numbers of the first kind.

Most of the other formulas in this article have a similar shape.

Setting in the above formula for $\sum_{k=0}^{\lfloor x \rfloor} \sqrt{k}$ the variable $x := n \in \mathbb{N}$, we obtain

$$\begin{aligned} \sum_{k=0}^n \sqrt{k} &= \frac{2}{3}n^{\frac{3}{2}} + \frac{1}{2}\sqrt{n} - \frac{1}{4\pi}\zeta\left(\frac{3}{2}\right) + \sqrt{n} \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^k (-1)^l \frac{(2l-3)!!}{2^l(l+1)!} B_{l+1} S_k^{(1)}(l)}{(n+1)(n+2)\cdots(n+k)} \\ &= \frac{2}{3}n^{\frac{3}{2}} + \frac{1}{2}\sqrt{n} - \frac{1}{4\pi}\zeta\left(\frac{3}{2}\right) + \frac{\sqrt{n}}{24(n+1)} + \frac{\sqrt{n}}{24(n+1)(n+2)} + \frac{53\sqrt{n}}{640(n+1)(n+2)(n+3)} \\ &\quad + \frac{79\sqrt{n}}{320(n+1)(n+2)(n+3)(n+4)} + \dots, \end{aligned}$$

which is the corresponding formula given in our previous paper [8].

2 Definitions

As usual, we denote the floor of x by $\lfloor x \rfloor$ and the fractional part of x by $\{x\}$.

Definition 1. (Pochhammer symbol)[1]

We define the *Pochhammer symbol* (or rising factorial function) $(x)_k$ by

$$(x)_k := x(x+1)(x+2)(x+3) \cdots (x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)},$$

where $\Gamma(x)$ is the gamma function defined by

$$\Gamma(x) := \int_0^\infty e^{-t} t^{x-1} dt.$$

Definition 2. (Stirling numbers of the first kind)[1]

Let $k, l \in \mathbb{N}_0$ be two non-negative integers such that $k \geq l \geq 0$. We define the *Stirling numbers of the first kind* $S_k^{(1)}(l)$ as the connecting coefficients in the identity

$$(x)_k = (-1)^k \sum_{l=0}^k (-1)^l S_k^{(1)}(l) x^l,$$

where $(x)_k$ is the rising factorial function.

Definition 3. (Bernoulli numbers)[2]

We define the k -th *Bernoulli number* B_k as the k -th coefficient in the generating function relation

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k \quad \forall x \in \mathbb{C} \text{ with } |x| < 2\pi.$$

Definition 4. (Euler numbers)[3]

We define the sequence of *Euler numbers* $\{E_k\}_{k=0}^{\infty}$ by the generating function identity

$$\frac{2e^x}{e^{2x} + 1} = \sum_{k=0}^{\infty} \frac{E_k}{k!} x^k.$$

Definition 5. (Bernoulli polynomials)[2, 3]

We define for $n \in \mathbb{N}_0$ the n -th *Bernoulli polynomial* $B_n(x)$ via the following exponential generating function as

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n \quad \forall t \in \mathbb{C} \text{ with } |t| < 2\pi.$$

Using this expression, we see that the value of the n -th Bernoulli polynomial $B_n(x)$ at the point $x = 0$ is

$$B_n(0) = B_n,$$

which is the n -th Bernoulli number.

Definition 6. (Euler polynomials)[3]

We define for $n \in \mathbb{N}_0$ the n -th *Euler polynomial* $E_n(x)$ via the following exponential generating function as

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} \frac{E_n(x)}{n!} t^n.$$

Moreover, we have that [4]

$$E_n(0) = -2(2^{n+1} - 1) \frac{B_{n+1}}{n+1} \quad \forall n \in \mathbb{N}_0.$$

3 Extended Summation Formulas involving Stirling Series

In this section we will prove our summation formulas for various finite sums of the form $\sum_{k=1}^{\lfloor x \rfloor} f(k)$. For this, we need the following

Lemma 7. (*Extended Euler-Maclaurin summation formula*)[3, 5]

Let f be an analytic function. Then for all $x \in \mathbb{R}^+$, we have that

$$\begin{aligned} \sum_{k=1}^{\lfloor x \rfloor} f(k) &= \int_1^x f(t) dt + \sum_{k=1}^m (-1)^k \frac{B_k(\{x\})}{k!} f^{(k-1)}(x) - \sum_{k=1}^m \frac{B_k}{k!} f^{(k-1)}(1) \\ &\quad + \frac{(-1)^{m+1}}{m!} \int_1^x B_m(\{t\}) f^{(m)}(t) dt, \end{aligned}$$

where $B_m(x)$ is the m -th Bernoulli polynomial and $\{x\}$ denotes the fractional part of x . Therefore, for many functions f we have the asymptotic expansion

$$\sum_{k=1}^{\lfloor x \rfloor} f(k) \sim \int_1^x f(t) dt + C + \sum_{k=1}^{\infty} (-1)^k \frac{B_k(\{x\})}{k!} f^{(k-1)}(x) \quad \text{as } x \rightarrow \infty,$$

for some constant $C \in \mathbb{C}$

and

Lemma 8. (*Extended Boole summation formula*)[3]

Let f be an analytic function. Then for all $x \in \mathbb{R}^+$, we have that

$$\begin{aligned} \sum_{k=1}^{\lfloor x \rfloor} (-1)^{k+1} f(k) &= \frac{(-1)^{x-\{x\}}}{2} \sum_{k=0}^m (-1)^{k+1} \frac{E_k(\{x\})}{k!} f^{(k)}(x) - \sum_{k=0}^m \frac{(2^{k+1} - 1)B_{k+1}}{(k+1)!} f^{(k)}(1) \\ &\quad + \frac{(-1)^m}{2m!} \int_1^x (-1)^{t-\{t\}} E_m(\{t\}) f^{(m+1)}(t) dt, \end{aligned}$$

where $E_m(x)$ is the m -th Euler polynomial and $\{x\}$ denotes the fractional part of x . Therefore, for many functions f we have the asymptotic expansion

$$\sum_{k=1}^{\lfloor x \rfloor} (-1)^{k+1} f(k) \sim C + \frac{(-1)^{x-\{x\}}}{2} \sum_{k=0}^{\infty} (-1)^{k+1} \frac{E_k(\{x\})}{k!} f^{(k)}(x) \quad \text{as } x \rightarrow \infty,$$

for some constant $C \in \mathbb{C}$.

From the above lemma we get by setting $x := n \in \mathbb{N}$ the following

Corollary 9. (Boole Summation Formula)[3]

Let f be an analytic function. Then for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{k=1}^n (-1)^{k+1} f(k) &= (-1)^n \sum_{k=0}^m (-1)^k \frac{(2^{k+1} - 1)B_{k+1}}{(k+1)!} f^{(k)}(n) - \sum_{k=0}^m \frac{(2^{k+1} - 1)B_{k+1}}{(k+1)!} f^{(k)}(1) \\ &\quad + \frac{(-1)^m}{2m!} \int_1^x (-1)^{t-\{t\}} E_m(\{t\}) f^{(m+1)}(t) dt, \end{aligned}$$

and the following asymptotic expansion

$$\sum_{k=1}^n (-1)^{k+1} f(k) \sim C + (-1)^n \sum_{k=0}^{\infty} (-1)^k \frac{(2^{k+1} - 1)B_{k+1}}{(k+1)!} f^{(k)}(n) \quad \text{as } n \rightarrow \infty,$$

for some constant $C \in \mathbb{C}$.

As well as the next key result found by J. Weniger:

Lemma 10. (Generalized Weniger transformation)[1]

For every inverse power series $\sum_{k=1}^{\infty} \frac{a_k(x)}{x^{k+1}}$, where $a_k(x)$ is any function in x , the following transformation formula holds

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{a_k(x)}{x^{k+1}} &= \sum_{k=1}^{\infty} \frac{(-1)^k}{(x)_{k+1}} \sum_{l=1}^k (-1)^l S_k^{(1)}(l) a_l(x) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k \sum_{l=1}^k (-1)^l S_k^{(1)}(l) a_l(x)}{x(x+1)(x+2) \cdots (x+k)}. \end{aligned}$$

Now, we prove as an example the following

Theorem 11. (Extended summation formulas for the harmonic series)

For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} = \log(x) + \gamma - \frac{B_1(\{x\})}{x} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k \frac{(-1)^l}{l+1} S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}.$$

We also have that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} = \log(x) + \gamma + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k \frac{(-1)^l}{l} S_k^{(1)}(l) B_l(\{x\})}{(x+1)(x+2) \cdots (x+k)}.$$

Proof. Applying the extended Euler-Maclaurin summation formula to the function $f(x) := \frac{1}{x}$, we get that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} \sim \log(x) + \gamma - \sum_{k=1}^{\infty} \frac{B_k(\{x\})}{k x^k}.$$

Applying now the generalized Weniger transformation to the equivalent series

$$\begin{aligned} \sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} &\sim \log(x) + \gamma - \frac{B_1(\{x\})}{x} - \sum_{k=2}^{\infty} \frac{B_k(\{x\})}{k x^k} \\ &\sim \log(x) + \gamma - \frac{B_1(\{x\})}{x} - \sum_{k=1}^{\infty} \frac{B_{k+1}(\{x\})}{(k+1) x^{k+1}}, \end{aligned}$$

we get the first claimed formula. To obtain the second expression, we apply the generalized Weniger transformation to the identity

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} \sim \log(x) + \gamma - x \sum_{k=1}^{\infty} \frac{B_k(\{x\})}{k x^{k+1}}.$$

□

At this point, we want to give an overview on summation formulas obtained with this method:

1.) **Extended summation formulas for the harmonic series:**

For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} = \log(x) + \gamma - \frac{B_1(\{x\})}{x} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k \frac{(-1)^l}{l+1} S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}$$

and

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} = \log(x) + \gamma + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k \frac{(-1)^l}{l} S_k^{(1)}(l) B_l(\{x\})}{(x+1)(x+2) \cdots (x+k)}.$$

2.) **Extended summation formulas for the partial sums of $\zeta(2)$:**

For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k^2} = \zeta(2) - \frac{1}{x} - \frac{B_1(\{x\})}{x^2} + \frac{1}{x} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k (-1)^l S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}$$

and

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k^2} = \zeta(2) - \frac{1}{x} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k (-1)^l S_k^{(1)}(l) B_l(\{x\})}{x(x+1)(x+2) \cdots (x+k)}.$$

3.) **Extended summation formulas for the partial sums of $\zeta(3)$:**

For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k^3} = \zeta(3) - \frac{1}{2x^2} - \frac{B_1(\{x\})}{x^3} + \frac{1}{2x^2} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k (-1)^l (l+2) S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}$$

and

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k^3} = \zeta(3) - \frac{1}{2x^2} + \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k (-1)^l (l+1) S_k^{(1)}(l) B_l(\{x\})}{x^2(x+1)(x+2) \cdots (x+k)}.$$

4.) **Extended summation formulas for the sum of the square roots:**

For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=0}^{\lfloor x \rfloor} \sqrt{k} = \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{4\pi} \zeta\left(\frac{3}{2}\right) - \sqrt{x} B_1(\{x\}) + \sqrt{x} \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^k \frac{(-1)^l (2l-3)!!}{2^l (l+1)!} S_k^{(1)}(l) B_{l+1}(\{x\})}{(x+1)(x+2) \cdots (x+k)},$$

$$\sum_{k=0}^{\lfloor x \rfloor} \sqrt{k} = \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{4\pi} \zeta\left(\frac{3}{2}\right) - \sqrt{x} B_1(\{x\}) + \frac{B_2(\{x\})}{4\sqrt{x}} + \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^k \frac{(-1)^l (2l-1)!!}{2^{l+1} (l+2)!} S_k^{(1)}(l) B_{l+2}(\{x\})}{\sqrt{x}(x+1)(x+2) \cdots (x+k)}$$

and

$$\sum_{k=0}^{\lfloor x \rfloor} \sqrt{k} = \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{4\pi} \zeta\left(\frac{3}{2}\right) + x\sqrt{x} \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^k \frac{(-1)^l (2l-5)!!}{2^{l-1} l!} S_k^{(1)}(l) B_l(\{x\})}{(x+1)(x+2) \cdots (x+k)}.$$

5.) **Extended summation formulas for the partial sums of $\zeta(-3/2)$:**

For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=0}^{\lfloor x \rfloor} k\sqrt{k} = \frac{2}{5}x^{\frac{5}{2}} - \frac{3}{16\pi^2}\zeta\left(\frac{5}{2}\right) - x^{\frac{3}{2}}B_1(\{x\}) + \frac{3}{2}x^{\frac{3}{2}}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sum_{l=1}^k\frac{(-1)^l(2l-5)!!}{2^{l-1}(l+1)!}S_k^{(1)}(l)B_{l+1}(\{x\})}{(x+1)(x+2)\cdots(x+k)},$$

$$\begin{aligned}\sum_{k=0}^{\lfloor x \rfloor} k\sqrt{k} &= \frac{2}{5}x^{\frac{5}{2}} - \frac{3}{16\pi^2}\zeta\left(\frac{5}{2}\right) - x^{\frac{3}{2}}B_1(\{x\}) + \frac{3}{4}\sqrt{x}B_2(\{x\}) - \frac{B_3(\{x\})}{4\sqrt{x}} \\ &\quad + \frac{3}{2}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sum_{l=1}^k\frac{(-1)^l(2l-1)!!}{2^{l+1}(l+3)!}S_k^{(1)}(l)B_{l+3}(\{x\})}{\sqrt{x}(x+1)(x+2)\cdots(x+k)}\end{aligned}$$

and

$$\sum_{k=0}^{\lfloor x \rfloor} k\sqrt{k} = \frac{2}{5}x^{\frac{5}{2}} - \frac{3}{16\pi^2}\zeta\left(\frac{5}{2}\right) + \frac{3}{2}x^{\frac{5}{2}}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sum_{l=1}^k\frac{(-1)^l(2l-7)!!}{2^{l-2}l!}S_k^{(1)}(l)B_l(\{x\})}{(x+1)(x+2)\cdots(x+k)}.$$

6.) **Extended summation formulas for the partial sums of $\zeta(-5/2)$:**

For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=0}^{\lfloor x \rfloor} k^2\sqrt{k} = \frac{2}{7}x^{\frac{7}{2}} + \frac{15}{64\pi^3}\zeta\left(\frac{7}{2}\right) - x^{\frac{5}{2}}B_1(\{x\}) + \frac{15}{4}x^{\frac{5}{2}}\sum_{k=1}^{\infty}(-1)^k\frac{\sum_{l=1}^k\frac{(-1)^l(2l-7)!!}{2^{l-2}(l+1)!}S_k^{(1)}(l)B_{l+1}(\{x\})}{(x+1)(x+2)\cdots(x+k)},$$

$$\begin{aligned}\sum_{k=0}^{\lfloor x \rfloor} k^2\sqrt{k} &= \frac{2}{7}x^{\frac{7}{2}} + \frac{15}{64\pi^3}\zeta\left(\frac{7}{2}\right) - x^{\frac{5}{2}}B_1(\{x\}) + \frac{5}{4}x^{\frac{3}{2}}B_2(\{x\}) - \frac{5}{8}\sqrt{x}B_3(\{x\}) + \frac{5B_4(\{x\})}{64\sqrt{x}} \\ &\quad + \frac{15}{4}\sum_{k=1}^{\infty}(-1)^k\frac{\sum_{l=1}^k\frac{(-1)^l(2l-1)!!}{2^{l+1}(l+4)!}S_k^{(1)}(l)B_{l+4}(\{x\})}{\sqrt{x}(x+1)(x+2)\cdots(x+k)}\end{aligned}$$

and

$$\sum_{k=0}^{\lfloor x \rfloor} k^2\sqrt{k} = \frac{2}{7}x^{\frac{7}{2}} + \frac{15}{64\pi^3}\zeta\left(\frac{7}{2}\right) + \frac{15}{4}x^{\frac{7}{2}}\sum_{k=1}^{\infty}(-1)^k\frac{\sum_{l=1}^k\frac{(-1)^l(2l-9)!!}{2^{l-3}l!}S_k^{(1)}(l)B_l(\{x\})}{(x+1)(x+2)\cdots(x+k)}.$$

7.) **Extended summation formulas for the sum of the inverse square roots:**

For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{\sqrt{k}} = 2\sqrt{x} + \zeta\left(\frac{1}{2}\right) - \frac{B_1(\{x\})}{\sqrt{x}} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k \frac{(-1)^l (2l-1)!!}{2^l (l+1)!} S_k^{(1)}(l) B_{l+1}(\{x\})}{\sqrt{x}(x+1)(x+2) \cdots (x+k)}$$

and

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{\sqrt{k}} = 2\sqrt{x} + \zeta\left(\frac{1}{2}\right) + \sqrt{x} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k \frac{(-1)^l (2l-3)!!}{2^{l-1} l!} S_k^{(1)}(l) B_l(\{x\})}{(x+1)(x+2) \cdots (x+k)}.$$

8.) **Extended summation formulas for the partial sums of $\zeta(3/2)$:**

For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k\sqrt{k}} = \zeta\left(\frac{3}{2}\right) - \frac{2}{\sqrt{x}} - \frac{B_1(\{x\})}{x\sqrt{x}} + \frac{2}{\sqrt{x}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k \frac{(-1)^l (2l+1)!!}{2^{l+1} (l+1)!} S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}$$

and

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k\sqrt{k}} = \zeta\left(\frac{3}{2}\right) - \frac{2}{\sqrt{x}} + 2 \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k \frac{(-1)^l (2l-1)!!}{2^l l!} S_k^{(1)}(l) B_l(\{x\})}{\sqrt{x}(x+1)(x+2) \cdots (x+k)}.$$

9.) **Extended summation formulas for the partial sums of $\zeta(5/2)$:**

For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k^2\sqrt{k}} = \zeta\left(\frac{5}{2}\right) - \frac{2}{3x^{\frac{3}{2}}} - \frac{B_1(\{x\})}{x^2\sqrt{x}} + \frac{4}{3x\sqrt{x}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k \frac{(-1)^l (2l+3)!!}{2^{l+2} (l+1)!} S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}$$

and

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k^2\sqrt{k}} = \zeta\left(\frac{5}{2}\right) - \frac{2}{3x^{\frac{3}{2}}} + \frac{4}{3\sqrt{x}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k \frac{(-1)^l (2l+1)!!}{2^{l+1} l!} S_k^{(1)}(l) B_l(\{x\})}{x(x+1)(x+2) \cdots (x+k)}.$$

10.) **Extended Generalized Faulhaber Formulas:**

For every positive real number $x \in \mathbb{R}^+$ and for every complex number $m \in \mathbb{C} \setminus \{-1\}$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} k^m = \frac{1}{m+1} x^{m+1} + \zeta(-m) + \frac{x^{m+1}}{m+1} \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^k \binom{m+1}{l} S_k^{(1)}(l) B_l(\{x\})}{(x+1)(x+2) \cdots (x+k)}$$

and

$$\sum_{k=1}^{\lfloor x \rfloor} k^m = \frac{1}{m+1} x^{m+1} + \zeta(-m) - x^m B_1(\{x\}) + \frac{x^m}{m+1} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k \binom{m+1}{l+1} S_k^{(1)}(l) B_{l+1}(\{x\})}{(x+1)(x+2) \cdots (x+k)}$$

and

$$\begin{aligned} \sum_{k=1}^{\lfloor x \rfloor} k^m &= \frac{1}{m+1} x^{m+1} + \zeta(-m) + \frac{1}{m+1} \sum_{k=1}^{\lfloor m+1 \rfloor} (-1)^k \binom{m+1}{k} B_k(\{x\}) x^{m-k+1} \\ &\quad + (-1)^{\lceil m+1 \rceil} \frac{x^{m-\lceil m+1 \rceil+2}}{m+1} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k \binom{m+1}{l+\lceil m+1 \rceil-1} S_k^{(1)}(l) B_{l+\lceil m+1 \rceil-1}(\{x\})}{(x+1)(x+2) \cdots (x+k)}. \end{aligned}$$

11.) Generalized Faulhaber Formulas:

For every complex number $m \in \mathbb{C} \setminus \{-1\}$ and every natural number $n \in \mathbb{N}$, we have that

$$\sum_{k=1}^n k^m = \frac{1}{m+1} n^{m+1} + \zeta(-m) + \frac{n^{m+1}}{m+1} \sum_{k=1}^{\infty} \frac{(-1)^k \sum_{l=1}^k \binom{m+1}{l} B_l S_k^{(1)}(l)}{(n+1)(n+2) \cdots (n+k)}$$

and

$$\sum_{k=1}^n k^m = \frac{1}{m+1} n^{m+1} + \zeta(-m) + \frac{1}{2} n^m + \frac{n^m}{m+1} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k \binom{m+1}{l+1} B_{l+1} S_k^{(1)}(l)}{(n+1)(n+2) \cdots (n+k)}$$

and

$$\begin{aligned} \sum_{k=1}^n k^m &= \frac{1}{m+1} n^{m+1} + \zeta(-m) + \frac{1}{m+1} \sum_{k=1}^{\lfloor m+1 \rfloor} (-1)^k \binom{m+1}{k} B_k n^{m-k+1} \\ &\quad + (-1)^{\lceil m+1 \rceil} \frac{n^{m-\lceil m+1 \rceil+2}}{m+1} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k \binom{m+1}{l+\lceil m+1 \rceil-1} B_{l+\lceil m+1 \rceil-1} S_k^{(1)}(l)}{(n+1)(n+2) \cdots (n+k)}. \end{aligned}$$

12.) Extended convergent versions of Stirling's formula:

For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} \log(k) = x \log(x) - x + \frac{1}{2} \log(2\pi) - \log(x) B_1(\{x\}) + \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^k \frac{(-1)^l}{l(l+1)} S_k^{(1)}(l) B_{l+1}(\{x\})}{(x+1)(x+2) \cdots (x+k)},$$

$$\begin{aligned} \sum_{k=1}^{\lfloor x \rfloor} \log(k) &= x \log(x) - x + \frac{1}{2} \log(2\pi) - \log(x) B_1(\{x\}) + \frac{B_2(\{x\})}{2x} \\ &\quad + \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^k \frac{(-1)^l}{(l+1)(l+2)} S_k^{(1)}(l) B_{l+2}(\{x\})}{x(x+1)(x+2) \cdots (x+k)} \end{aligned}$$

and that

$$\sum_{k=1}^{\lfloor x \rfloor} \log(k) = x \log(x) - x + \frac{1}{2} \log(2\pi) - \log(x) B_1(\{x\}) + x \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=2}^k \frac{(-1)^l}{l(l-1)} S_k^{(1)}(l) B_l(\{x\})}{(x+1)(x+2) \cdots (x+k)}.$$

13.) **Extended first logarithmic summation formulas:**

For every positive real number $x \in \mathbb{R}^+$, we have that

$$\begin{aligned} \sum_{k=0}^{\lfloor x \rfloor} k \log(k) &= \frac{1}{2} x^2 \log(x) - \frac{1}{4} x^2 + \frac{1}{12} - \zeta'(-1) - x \log(x) B_1(\{x\}) + \frac{1}{2} \log(x) B_2(\{x\}) \\ &\quad + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k \frac{(-1)^l}{l(l+1)(l+2)} S_k^{(1)}(l) B_{l+2}(\{x\})}{(x+1)(x+2) \cdots (x+k)} \end{aligned}$$

and

$$\begin{aligned} \sum_{k=0}^{\lfloor x \rfloor} k \log(k) &= \frac{1}{2} x^2 \log(x) - \frac{1}{4} x^2 + \frac{1}{12} - \zeta'(-1) - x \log(x) B_1(\{x\}) + \frac{1}{2} \log(x) B_2(\{x\}) - \frac{B_3(\{x\})}{6x} \\ &\quad + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k \frac{(-1)^l}{(l+1)(l+2)(l+3)} S_k^{(1)}(l) B_{l+3}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}. \end{aligned}$$

14.) **Extended second logarithmic summation formula:**

For every positive real number $x \in \mathbb{R}^+$, we have that

$$\begin{aligned} \sum_{k=1}^{\lfloor x \rfloor} \frac{\log(k)}{k} &= \frac{1}{2} \log(x)^2 + \gamma_1 - \frac{\log(x)}{x} B_1(\{x\}) + \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^k \frac{(-1)^l}{(l+1)!} S_{l+1}^{(1)}(2) S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+1) \cdots (x+k)} \\ &\quad + \log(x) \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^k \frac{1}{l+1} S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}. \end{aligned}$$

15.) **Extended third logarithmic summation formula:**

For every positive real number $x \in \mathbb{R}^+$, we have that

$$\begin{aligned} \sum_{k=1}^{\lfloor x \rfloor} \frac{\log(k)}{k^2} &= -\zeta'(2) - \frac{\log(x)}{x} - \frac{1}{x} - \frac{\log(x)}{x^2} B_1(\{x\}) \\ &+ \frac{1}{x} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^k \frac{1}{l+1} \left(\sum_{m=0}^{l-1} \frac{m+1}{l-m} \right) S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2) \cdots (x+k)} \\ &+ \frac{\log(x)}{x} \sum_{k=1}^{\infty} \frac{(-1)^k \sum_{l=1}^k S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}. \end{aligned}$$

16.) **Extended fourth logarithmic summation formula:**

For every positive real number $x \in \mathbb{R}^+$, we have that

$$\begin{aligned} \sum_{k=1}^{\lfloor x \rfloor} \log(k)^2 &= x \log(x)^2 - 2x \log(x) + 2x + \frac{\gamma^2}{2} - \frac{\pi^2}{24} - \frac{\log(2)^2}{2} - \log(2) \log(\pi) - \frac{\log(\pi)^2}{2} + \gamma_1 \\ &- \log(x)^2 B_1(\{x\}) + \frac{\log(x)}{x} B_2(\{x\}) + 2 \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^k \frac{(-1)^l}{(l+2)!} S_{l+1}^{(1)}(2) S_k^{(1)}(l) B_{l+2}(\{x\})}{x(x+1)(x+2) \cdots (x+k)} \\ &+ 2 \log(x) \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^k \frac{1}{(l+1)(l+2)} S_k^{(1)}(l) B_{l+2}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}. \end{aligned}$$

17.) **Extended summation formula for partial sums of the Gregory-Leibniz series:**

For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=0}^{\lfloor x \rfloor} \frac{(-1)^k}{2k+1} = \frac{\pi}{4} + \frac{(-1)^{x-\{x\}}}{2} \sum_{k=0}^{\infty} \frac{(-1)^k \sum_{l=0}^k (-1)^l 2^l S_k^{(1)}(l) E_l(\{x\})}{(2x+1)(2x+2)(2x+3) \cdots (2x+k+1)}.$$

Setting $x := n \in \mathbb{N}$, we get that

$$\sum_{k=0}^n \frac{(-1)^k}{2k+1} = \frac{\pi}{4} + (-1)^{n+1} \sum_{k=0}^{\infty} \frac{(-1)^k \sum_{l=0}^k \frac{(-1)^l}{l+1} 2^l (2^{l+1} - 1) S_k^{(1)}(l) B_{l+1}}{(2n+1)(2n+2)(2n+3) \cdots (2n+k+1)}.$$

18.) **Extended formula for the partial sums of the alternating harmonic series:**

For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{(-1)^{k+1}}{k} = \log(2) + \frac{(-1)^{x-\{x\}}}{2} \sum_{k=0}^{\infty} (-1)^{k+1} \frac{\sum_{l=0}^k (-1)^l S_k^{(1)}(l) E_l(\{x\})}{x(x+1)(x+2) \cdots (x+k)}.$$

Setting $x := n \in \mathbb{N}$, we get

$$\sum_{k=1}^n \frac{(-1)^{k+1}}{k} = \log(2) + (-1)^n \sum_{k=0}^{\infty} (-1)^k \frac{\sum_{l=0}^k \frac{(-1)^l}{l+1} (2^{l+1} - 1) S_k^{(1)}(l) B_{l+1}}{n(n+1)(n+2) \cdots (n+k)}.$$

4 Other Extended Summation Formulas for Finite Sums

In this section, we denote by $\eta(s) := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^s}$ the Dirichlet eta function.

1.) **The extended alternating Faulhaber formula:**

$$\sum_{k=1}^{\lfloor x \rfloor} (-1)^{k+1} k^m = \eta(-m) + \frac{(-1)^{x-\{x\}}}{2(m+1)} \sum_{k=0}^m (-1)^{k+1} (k+1) \binom{m+1}{k+1} E_k(\{x\}) x^{m-k} \quad \forall m \in \mathbb{N}_0.$$

Setting $x := n \in \mathbb{N}$, we get

$$\sum_{k=1}^n (-1)^{k+1} k^m = \eta(-m) + \frac{(-1)^n}{m+1} \sum_{k=0}^m (-1)^k (2^{k+1} - 1) \binom{m+1}{k+1} B_{k+1} n^{m-k} \quad \forall m \in \mathbb{N}_0.$$

2.) **The extended generalized alternating Faulhaber formula:**

$$\sum_{k=1}^{\lfloor x \rfloor} (-1)^{k+1} k^m = \eta(-m) + \frac{(-1)^{x-\{x\}} x^m}{2(m+1)} \sum_{k=0}^{\infty} (-1)^{k+1} \frac{\sum_{l=0}^k (l+1) \binom{m+1}{l+1} S_k^{(1)}(l) E_l(\{x\})}{(x+1)(x+2) \cdots (x+k)} \quad \forall m \in \mathbb{C}.$$

Setting $x := n \in \mathbb{N}$, we get

$$\sum_{k=1}^n (-1)^{k+1} k^m = \eta(-m) + \frac{(-1)^n n^m}{(m+1)} \sum_{k=0}^{\infty} (-1)^k \frac{\sum_{l=0}^k \binom{m+1}{l+1} (2^{l+1} - 1) S_k^{(1)}(l) B_{l+1}}{(n+1)(n+2) \cdots (n+k)} \quad \forall m \in \mathbb{C}.$$

3.) **The Geometric Summation Formula:**

$$\sum_{k=0}^{\lfloor x \rfloor} a^k = \frac{a^x}{\log(a)} + \frac{1}{1-a} + a^x \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^k \frac{\log(a)^{l-1}}{l!} S_k^{(1)}(l) B_l(\{x\}) x^l}{(x+1)(x+2) \cdots (x+k)} \quad \forall a \neq 1.$$

Setting $x := n \in \mathbb{N}$, we get

$$\sum_{k=0}^n a^k = \frac{a^n}{\log(a)} + \frac{1}{1-a} + a^n \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^k \frac{\log(a)^{l-1}}{l!} S_k^{(1)}(l) B_l n^l}{(n+1)(n+2) \cdots (n+k)} \quad \forall a \neq 1.$$

4.) **The alternating Geometric Summation Formula:**

$$\sum_{k=0}^{\lfloor x \rfloor} (-1)^k a^k = \frac{1}{1+a} + \frac{(-1)^{x-\{x\}}}{2} a^x \sum_{k=0}^{\infty} (-1)^k \frac{\sum_{l=0}^k \frac{\log(a)^l}{l!} S_k^{(1)}(l) E_l(\{x\}) x^l}{(x+1)(x+2) \cdots (x+k)} \quad \forall a \neq -1.$$

Setting $x := n \in \mathbb{N}$, we get

$$\sum_{k=0}^n (-1)^k a^k = \frac{1}{1+a} + (-1)^{n+1} a^n \sum_{k=0}^{\infty} (-1)^k \frac{\sum_{l=0}^k \frac{\log(a)^l}{(l+1)!} (2^{l+1} - 1) S_k^{(1)}(l) B_{l+1} n^l}{(n+1)(n+2) \cdots (n+k)} \quad \forall a \neq -1.$$

5.) **The Euler-Maclaurin Geometric Summation Formula:**

$$\sum_{k=0}^{\lfloor x \rfloor} a^k = \frac{a^x}{\log(a)} + \frac{1}{1-a} + a^x \sum_{k=1}^{\infty} (-1)^k \frac{\log(a)^{k-1}}{k!} B_k(\{x\}) \quad \text{for } \frac{1}{e^{2\pi}} < a \neq 1 < e^{2\pi}.$$

6.) **The Euler-Maclaurin alternating Geometric Summation Formula:**

$$\sum_{k=0}^{\lfloor x \rfloor} (-1)^k a^k = \frac{1}{1+a} + (-1)^{x-\{x\}} \frac{a^x(a-1)}{a+1} \sum_{k=0}^{\infty} (-1)^k \frac{\log(a)^{k-1}}{k!} B_k(\{x\}) \quad \text{for } \frac{1}{e^{2\pi}} < a < e^{2\pi}.$$

7.) **The Exponential Geometric Summation Formula:**

$$\sum_{k=0}^{\lfloor x \rfloor} e^k = e^x + \frac{1}{1-e} + e^x \sum_{k=1}^{\infty} (-1)^k \frac{B_k(\{x\})}{k!} \quad \forall x \in \mathbb{R}^+.$$

8.) **The Self-Counting Summation Formula:**

Let $\{a_k\}_{k=1}^{\infty} := \{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \dots\}$ be the self-counting sequence [6, 7] defined by

$$a_k := \left\lfloor \frac{1}{2} + \sqrt{2k} \right\rfloor.$$

Then, we have that

$$\begin{aligned} \sum_{k=1}^{\lfloor x \rfloor} a_k &= \frac{x\sqrt{8x+1}}{3} - \frac{5\sqrt{8x+1}}{24} - \frac{\sqrt{8x+1}}{2} B_1(\{x\}) + B_1 \left(\left\{ \frac{\sqrt{8x+1}}{2} - \frac{1}{2} \right\} \right) B_1(\{x\}) \\ &\quad + \frac{1}{2} B_1 \left(\left\{ \frac{\sqrt{8x+1}}{2} - \frac{1}{2} \right\} \right) - \frac{\sqrt{8x+1}}{4} B_2 \left(\left\{ \frac{\sqrt{8x+1}}{2} - \frac{1}{2} \right\} \right) \\ &\quad + \frac{1}{6} B_3 \left(\left\{ \frac{\sqrt{8x+1}}{2} - \frac{1}{2} \right\} \right). \end{aligned}$$

9.) **Slowly convergent summation formula for the sum of the square roots:**

$$\sum_{k=0}^{\lfloor x \rfloor} \sqrt{k} = \frac{2}{3}x^{\frac{3}{2}} - \sqrt{x}B_1(\{x\}) - \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{\text{FresnelS}\left(2\sqrt{k}\sqrt{x}\right)}{k^{\frac{3}{2}}}.$$

10.) **Slowly convergent summation formula for the harmonic series:**

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} = \log(x) + \gamma + 2 \sum_{k=1}^{\infty} \text{CosIntegral}(2\pi kx).$$

5 Conclusion

This paper presents an overview on some rapidly convergent summation formulas obtained by applying the Weniger transformation [1].

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